IV Alternative models

* The informal notion of $(\infty, 1)$-category admits many different formal incarnations, or models.

* This diversity of models has some advantages :
  - they illuminate different aspects of the theory.
  - different examples are more naturally presented in different models.

* But mostly it is a major headache for the theory :
  - imagine sitting through 4 or 5 courses like this, developping each model from scratch!

- we mostly do not care about the specific features of this or that model.

* 3 classes of solutions to this problem:

- Linguistic / Pragmatic:

Work within a specific model, but adopt terminology and notations that deemphasize the particular features and can be relatively easily transcribed into another by someone who knows both sides well.

This is the main approach in practice so far, with the chosen model being almost always quasicategories.

- Comparative / synthetic:

Develop axiomatic frameworks for "models of ∞-categories" which capture features of the theory formally, and prove results within that framework.

This has been done for the "homotopy theory of $(\infty,1)$-categories" (Toen) and for the "2-category theory of $(\infty,1)$-categories" (Riehl-Verity, <u>Elements of ∞-category theory</u>)

- <u>Foundational:</u>

Rewrite the foundations of mathematics by "replacing sets by ∞-groupoids."
Then $(\infty,1)$-categories can be built up directly from ∞-groupoids.

There is such an approach with homotopy type theory. An implementation of $(\infty,1)$-categories in this context is in Riehl-Shulman,

A type theory for synthetic ∞-categories.

* To convince you that it is a problem that needs solving, I need to present some other models!

More concretely, this will also give us techniques to construct many new quasicategories.

1) Enriched categories

def 1: A monoidal category is a category V together with a monoidal structure:

- a tensor product $-\otimes- : V \times V \longrightarrow V$
- a unit object $\mathbb{1} \in V$
- left and right unitors $\begin{cases} \eta_\ell : id_V \xrightarrow{\sim} \mathbb{1} \otimes id_V \\ \eta_r : id_V \longrightarrow id_V \otimes \mathbb{1} \end{cases}$

- an associator $\alpha : (- \otimes -) \otimes - \xrightarrow{\sim} - \otimes (- \otimes -)$

such that the following diagrams commute:

- (triangle identity)

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "\alpha"] \arrow[dr, "\eta_r \otimes id"'] & \arrow[d, phantom, "\equiv"] & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "id \otimes \eta_\ell"] \\
& X \otimes Y &
\end{tikzcd}$$

- (pentagon identity)

$$\begin{tikzcd}
& (X \otimes Y) \otimes (Z \otimes W) \arrow[dr, "\alpha"] & \\
((X \otimes Y) \otimes Z) \otimes W \arrow[ur, "\alpha"] \arrow[d, "\alpha \otimes id"'] & \equiv & X \otimes (Y \otimes (Z \otimes W)) \\
((X \otimes (Y \otimes Z)) \otimes W \arrow[rr, "\alpha"] & & (X \otimes ((Y \otimes Z) \otimes W)) \arrow[u, "id \otimes \alpha"']
\end{tikzcd}$$

Ex 2 * Any category with finite products has the cartesian monoidal structure with

$\otimes = \times$ and $\mathbb{1}$ = terminal object.

$\rightsquigarrow$ Set, sSet, Cat, Top, ...

* Let R be a comm. ring. The category Mod_R of R-modules, equipped with $\otimes = \otimes_R$ and $\mathbb{1} = R$, is a monoidal category.

* Let Cpl_R be the category of chain complexes of R-modules. Then the usual tensor product of chain complexes induces a monoidal structure with unit $R[0]$.

* (Monoidal categories with one object) $\simeq$ (Monoids)

$$(\mathbb{1}, \mathbb{1}, *) \longmapsto \mathrm{End}(\mathbb{1})$$

... (many other examples)

(In the rest of this section, we always equip these examples with those monoidal structures.)

RmK 3

- The axioms are precisely designed so that we can manipulate $\otimes$ and $\mathbb{1}$ as we are used to in the previous examples, i.e., as if η_e, η_r and α were "essentially identities" (this can be made formal via Maclane's coherence theorem).

- All the previous examples (except monoids) admit a further **symmetric monoidal structure**: a natural isomorphism $\tau : X \otimes Y \simeq Y \otimes X$ satisfying $\tau^2 = id$ and a further "hexagon identity".

This is of course very important, but not necessary for enriched categories.

(Natural examples of non-symmetric monoidal categories: (R,R)-bimodules for R non-comm ring, left modules over an Hopf algebra ...)

def 4: Let $(V, \otimes, \mathbb{1})$ be a monoidal category. A **V-enriched category** C consists of

- a collection $Ob(C)$ of objects.

- for any two $X, Y \in Ob(C)$, an object
  $\mathrm{Hom}_C(X,Y) \in V$.

- For any $X, Y, Z \in Ob(C)$, a "composition"

$$\circ : \mathrm{Hom}_C(X,Y) \otimes \mathrm{Hom}_C(Y,Z) \longrightarrow \mathrm{Hom}_C(X,Z).$$

- For any $X \in Ob(C)$, an "identity"

$$id_X : \mathbb{1} \longrightarrow \mathrm{Hom}_C(X,X).$$

satisfying associativity and unitality conditions:

$$\begin{tikzcd}
\left(\mathrm{Hom}_C(X,Y) \otimes \mathrm{Hom}_C(Y,Z)\right) \otimes \mathrm{Hom}_C(Z,W) \arrow[r, "\circ \otimes id"] \arrow[dd, "\alpha"'] \arrow[ddr, phantom, "\equiv"] & \mathrm{Hom}_C(X,Z) \otimes \mathrm{Hom}_C(Z,W) \arrow[d, "\circ"] \\
 & \mathrm{Hom}_C(X,W) \\
\mathrm{Hom}_C(X,Y) \otimes \left(\mathrm{Hom}_C(Y,Z) \otimes \mathrm{Hom}_C(Z,W)\right) \arrow[r] & \mathrm{Hom}_C(X,Y) \otimes \mathrm{Hom}_C(Y,W) \arrow[u, "\circ"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hom}_C(X,X) \otimes \mathrm{Hom}_C(X,Y) \arrow[r, "\circ"] & \mathrm{Hom}_C(X,Y) & \mathrm{Hom}_C(X,Y) \otimes \mathrm{Hom}_C(Y,Y) \arrow[l, "\circ"'] \\
\mathbb{1} \otimes \mathrm{Hom}_C(X,Y) \arrow[u, "\mathrm{id}_X"] \arrow[ur, "\eta_\ell"] \arrow[u, phantom, shift left=12ex, "\equiv"] & & \mathrm{Hom}_C(X,Y) \otimes \mathbb{1} \arrow[u, "\mathrm{id}_Y"'] \arrow[ul, "\eta_r"'] \arrow[u, phantom, shift right=12ex, "\equiv"]
\end{tikzcd}$$

- A V-enriched functor $F : C \to D$ between V-enriched categories consists of
  - a map $F : \mathrm{Ob}(C) \longrightarrow \mathrm{Ob}(D)$
  - for $X, Y \in \mathrm{Ob}(C)$, a morphism
    $$F_{X,Y} : \mathrm{Hom}_C(X,Y) \longrightarrow \mathrm{Hom}_D(F(X), F(Y))$$

  such that the diagrams

$$\begin{tikzcd}
\mathrm{Hom}_C(X,Y) \otimes \mathrm{Hom}_C(Y,Z) \arrow[r, "\circ"] \arrow[d, "F_{X,Y} \otimes F_{Y,Z}"'] \arrow[dr, phantom, "\equiv"] & \mathrm{Hom}_C(X,Z) \arrow[d, "F_{X,Z}"] \\
\mathrm{Hom}_D(F(X),F(Y)) \otimes \mathrm{Hom}_D(F(Y),F(Z)) \arrow[r] & \mathrm{Hom}_D(F(X),F(Z))
\end{tikzcd}$$

and
$$\begin{tikzcd}
\mathbb{1} \arrow[r, "id_X"] \arrow[dr, "id_{F(X)}"'] & \mathrm{Hom}_C(X,X) \arrow[d, "F_{X,X}"] \arrow[dl, phantom, "="] \\
& \mathrm{Hom}_D(F(X),F(X))
\end{tikzcd}$$

commute.

- The collection of all V-enriched categories and V-enriched functors forms a category Cat_V.

Ex 5:

- A Set-enriched category is just a category.
- An Ab-enriched category is sometimes called a pre-additive category (an additive category is an Ab-enriched category with finite products).

- A Mod_R-enriched category is an R-linear category.

- A sSet-enriched category is a simplicial category.

  (⚠ Possible confusion; this is different from a simplicial object in Cat.)

- A Top-enriched category is a topological category.

- A Cpl_R-enriched category is a (R-linear) differential graded category.

- A Cat-enriched category is a strict 2-category.

* We want to relate different types of enriched categories.

def 6: Let $(V, \otimes, \mathbb{1})$ and $(W, \otimes, \mathbb{1})$ be two monoidal categories. A lax monoidal functor $F : (V, \otimes, \mathbb{1}) \longrightarrow (W, \otimes, \mathbb{1})$ consists of:

- A functor $F : V \longrightarrow W$
- A morphism $\mathbb{1}_V \longrightarrow F(\mathbb{1}_W)$
- A natural transformation:

$$F(-) \otimes F(-) \longrightarrow F(- \otimes -)$$

satisfying some compatibility diagrams which I won't spell out.

An oplax monoidal functor is similar, but with

$$F(- \otimes -) \longrightarrow F(-) \otimes F(-).$$

A monoidal functor is a lax (or oplax) functor with

$$F(-) \otimes F(-) \xrightarrow{\sim} F(- \otimes -).$$

. A natural transformation $F \to G$ between lax-monoidal functors is called lax monoidal if

$$\begin{tikzcd} FX \otimes FY \arrow[r] \arrow[d] \arrow[dr, phantom, "\equiv"] & F(X \otimes Y) \arrow[d] \\ GX \otimes GY \arrow[r] & G(X \otimes Y) \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[d] \arrow[dr] \arrow[dr, phantom, "\equiv"] & \\ F(\mathbb{1}) \arrow[r] & G(\mathbb{1}) \end{tikzcd}$$

Ex 7:

* Let C, D be categories with finite products. $F : C \to D$ a product-preserving functor.

  Then F induces a monoidal functor

$$(C, \times, *) \longrightarrow (D, \times, *).$$

* Let V be any monoidal category. Then $\mathrm{Hom}(\mathbb{1}, -) : V \longrightarrow \mathrm{Set}$

has a canonical lax-monoidal structure (Exercise)
(can often be interpreted as the "underlying set" of V)

* Consider the trivial category $*$; it has finite products (!) so admit a (cartesian) monoidal structure. A lax-monoidal functor

$$* \longrightarrow V \quad \text{for } V \text{ monoidal is an}$$

algebra object in V: an object A

together with $\begin{cases} \text{a multiplication } m : A \otimes A \to A \\ \text{a unit } e : \mathbb{1} \longrightarrow A \end{cases}$

satisfying the usual monoid axioms.
(recovers monoids in Set, R-algebras in Mod_R, dg-R-algebras in Cpl_R, ...)

Lemma 7: Let $F : (V, \otimes, \mathbb{1}) \to (W, \otimes, \mathbb{1})$ be a lax-monoidal functor. There is an induced functor

$$F_* : Cat_V \longrightarrow Cat_W$$

with: $Ob(F_*(C)) = Ob(C)$

- $$Hom_{F_*(C)}(X, Y) = F(Hom_C(X, Y))$$

- Composition in $F_*(C)$ is obtained via the composition in C and the lax-monoidal structure.

- If F is fully faithful, then so is F_*.

$\square$

2) Simplicial categories

- Cat_{sSet} and Cat_{Top} provide two of the simplest models of $(\infty,1)$-categories.

- The story for the two is really parallel, and I will concentrate on $\mathrm{Cat}_\Delta := \mathrm{Cat}_{sSet}$. (see exercise sheet for Cat_{Top}).

  Unlike pretty much everything (about ∞-categories) discussed so far, this material is not due to Joyal / Lurie but to Cordier and Porter.

- We start by relating Cat_Δ to other types of categories. We have functors:

$*$ $c : \mathrm{Set} \longrightarrow s\mathrm{Set}$ constant simplicial set

$\mathrm{ev}_n : s\mathrm{Set} \longrightarrow \mathrm{Set}$ n-simplices

$\big(\overset{\shortparallel}{s\mathrm{Set}}(\Delta^n, -)\big)$

$\pi_0 : sSet \longrightarrow Set$ connected components

$N : Cat \longrightarrow sSet$ nerve

$|\cdot| : sSet \longrightarrow \mathcal{H}$ geometric realisation

homotopy category of spaces

$Sing : Top \longrightarrow sSet$ singular complex

They all preserve finite products (this is obvious for c and ev_n, N and $Sing$ are right adjoints, π_0 was an exercise on Sheet 3, the case of $|\cdot|$ was discussed in Lecture 4).

$\rightsquigarrow$ they have natural monoidal structures (with cartesian structures on Set, sSet, Cat)

def 8: By applying Lemma 7, we get

- $c := cat_* : Cat \xhookrightarrow{\delta.\delta.} Cat_\Delta$ constant simplicial category.

- $\pi := (\pi_0)_* : \mathrm{Cat}_\Delta \to \mathrm{Cat}$ (truncated) homotopy category
- $\pi_\infty := |\cdot|_* : \mathrm{Cat}_\Delta \to \mathrm{Cat}_{\mathcal{H}}$ (full) homotopy category
- $u := (ev_0)_* : \mathrm{Cat}_\Delta \to \mathrm{Cat}$ underlying category
- $N := N_* : \mathrm{Cat}_2 = \mathrm{Cat}_{\mathrm{Cat}} \overset{f.f.}{\hookrightarrow} \mathrm{Cat}_\Delta$ nerve (?)
- Since we have an adjunction $\pi_0 : s\mathrm{Set} \rightleftarrows \mathrm{Set} : c$, there is an induced adjunction:

$$\pi : \mathrm{Cat}_\Delta \rightleftarrows \mathrm{Cat} : c .$$

(this is another general fact of (enriched) life: monoidal adjunctions induce adjunctions of categories of enriched categories; no time for that now!)

def 9: Let $F : C \to D$ in Cat_Δ.

We say that F is a weak equivalence (or Dwyer-Kan equivalence) if one of the two following equivalent conditions hold:

$\ast$ $\begin{cases} \text{- For all } X, Y \in Ob(C), \quad Hom_C(X,Y) \to Hom_C(F(X), F(Y)) \\ \quad \text{is a weak equivalence of simplicial sets.} \\ \text{- The functor } \pi F : \pi C \to \pi D \text{ is essentially} \\ \quad \text{surjective.} \end{cases}$

$\ast$ The functor $\pi_\infty F$ is an isomorphism in $Cat_{\mathcal{H}}$.

def 10: A simplicial category C is locally Kan if for all $X, Y \in Ob(C)$, $Hom_C(X,Y)$ is a Kan complex. $\square$

3) Simplicial categories vs ∞-categories

We want to construct a version of the nerve functor which takes the simplicial enrichment into account.

$\rightsquigarrow$ we replace $[n]$ by an appropriate simplicial category.

def 11: Let J be a finite non-empty totally ordered set, and $i \leq j$ in J.

Let $P_{i,j} = \{ I \subseteq [i,j] \mid i,j \in J \}$,

considered as poset ordered by inclusion.

- Let $i \leq j \leq k$ in J. We define a map of posets

$$\begin{aligned} P_{i,j} \times P_{j,k} &\longrightarrow P_{i,k} \\ (I \, , \, I') &\longmapsto I \cup I'. \end{aligned}$$

- We define a simplicial category $\mathrm{Path}[\Delta^J] \in \mathrm{Cat}_\Delta$.

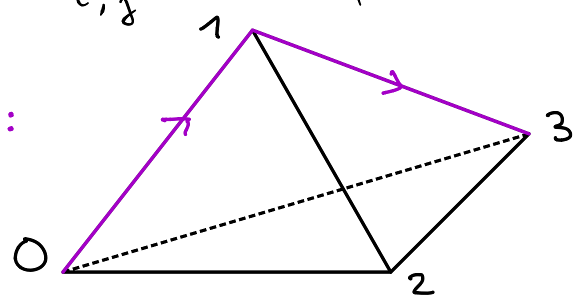
- $\mathrm{Ob}(\mathrm{Path}[\Delta^J]) = J$

- $$\mathrm{Hom}_{\mathrm{Path}[\Delta^J]}(i,j) = \begin{cases} \varnothing \, , & j < i \\ N(P_{i,j}) \, , & j \geq i \end{cases}$$

Composition is defined by the map above; identities are given by $\{i\} \in P_{i,i}$, and associativity follows from the associativity of $\cup$.

Rmk 12: An alternative notation, used in HTT, is $\mathfrak{C}[\Delta^J]$. The notation "Path" (Kerodon) records the idea that simplices in $P_{i,j}$ record paths in Δ^J:

An element in $P_{0,3}$:



Lemma 13:
- $N(P_{i,j}) \simeq (\Delta^1)^{|[i,j]|-2}$
- $N(P_{0,n}) \simeq (\Delta^1)^{n-1}$

(simplicial cubes.)

proof:
$$\begin{cases} P_{0,n} \simeq \mathcal{P}([1,n-1]) \simeq [1]^{\times(n-1)} \\ N \text{ commutes with products.} \end{cases}$$
$\square$

Lemma 14: There is a canonical isomorphism of categories $\pi\, \mathrm{Path}[\Delta^n] \simeq [n]$.

By adjunction, we get a functor

$$\mathrm{Path}[\Delta^n] \longrightarrow [n]$$

which is a weak equivalence of simplicial categories.

proof: exercise (Hint: cubes are contractible) $\square$

* So we have replaced $[n]$ by an equivalent simplicial category, which is "free" in some sense that can be made precise.

Ex 15: $\cdot$ $\mathrm{Path}[\Delta^0] = c[0]$
$\cdot$ $\mathrm{Path}[\Delta^1] = c[1]$ $\Big\}$ nothing happens for 0 and 1 simplices.

$\cdot$ $\mathrm{Path}[\Delta^2]$ can be depicted as

$$\begin{tikzcd}
 & 1 \arrow[dr] & \\
0 \arrow[ur] \arrow[rr] \arrow[rr, phantom, shift left=4ex, "\Uparrow"] & & 2
\end{tikzcd}$$

i.e in $\mathrm{Path}[\Delta^2](0,2) = P_{0,2} \simeq \Delta^1$, there is exactly one (non-deg.) morphism

$$\{0,2\} \longrightarrow \{1,2\} \circ \{0,1\} = \{0,1,2\}$$

So $\mathrm{Path}[\Delta^2]$ encodes a kind of universal "composition of 2-morphisms up to homotopy".

· The construction of $\mathrm{Path}[\Delta^J]$ is clearly functorial w.r.t order-preserving maps $J \to J'$

$\rightsquigarrow$ cosimplicial simplicial category (!)

$$\begin{aligned} \mathrm{Path}[\Delta^\bullet] : \Delta &\longrightarrow \mathrm{Cat}_\Delta \\ [n] &\longmapsto \mathrm{Path}[\Delta^n] \end{aligned}$$

def 16: (Cordier) The homotopy coherent nerve is the functor induced by $\mathrm{Path}[\Delta^\bullet]$:

$$\begin{aligned} N_\Delta : \mathrm{Cat}_\Delta &\longrightarrow \mathrm{sSet} \\ C &\longmapsto \left([n] \mapsto \mathrm{Cat}_\Delta(\mathrm{Path}[\Delta^n], C)\right) \end{aligned}$$

Lemma 17: The functors N and $N_\Delta \circ c$ are naturally isomorphic: $N_\Delta(cC) \simeq N(C)$ for C 1-category.

proof:
$$
\begin{aligned}
N(cC)_n &= \mathrm{Cat}_\Delta(\mathrm{Path}[\Delta^n], cC) \\
&\overset{\text{adj.}}{=} \mathrm{Cat}(\pi\,\mathrm{Path}[\Delta^n], C) \\
&\overset{\text{lemma 14}}{=} \mathrm{Cat}([n], C) \\
&= N(C)_n .
\end{aligned}
$$
$\square$

Ex 18: Let $C \in \mathrm{Cat}_\Delta$.

- $N(C)_0 = \mathrm{Ob}(C)$
- $N(C)_1 = \mathrm{Mor}(uC)$
- $N(C)_2 = \left\{ \begin{array}{l} (X,Y,Z) \in \mathrm{Ob}(C),\ f: X \to Y,\ g: Y \to Z \\ \text{and } \Delta^1 \xrightarrow{h} \mathrm{Hom}_C(X,Z) \text{ with } h_0 = g \circ f \end{array} \right\}$

$$
\begin{tikzcd}
 & Y \arrow[dr, "g"] \arrow[d, phantom, "\Uparrow h"] & \\
X \arrow[ur, "f"] \arrow[rr, "h_1"'] & {} & Z
\end{tikzcd}
$$

- The nerve of a category was automatically an ∞-category. This is not the case for simplicial categories (but it's not so easy to write an explicit counter-example). Rather, we have

Thm 19: (Cordier-Porter)

Let C be a locally Kan simplicial category. Then $N_\Delta(C)$ is an ∞-category.

The proof requires some preliminaries.

- First, I will admit the following fact for convenience (in practice, we will need some rather simple colimits.)

Prop 20: The category Cat_Δ is (complete and) cocomplete. $\square$

- Using this and the results from the very first lecture, we know that N_Δ has a left adjoint $\mathrm{Path}[-]$, which is the unique colimit-preserving functor

$$\mathrm{Path}[\cdot] : sSet \longrightarrow Cat_\Delta$$

extending the $\mathrm{Path}[\Delta^n]$.

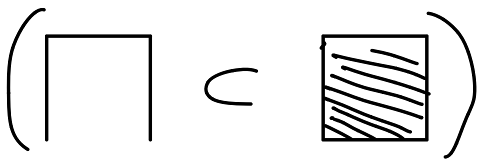
- <u>Lemma 21:</u> Let $0 < j < n$.

The simplicial category $\mathrm{Path}[\Lambda^n_j]$ is the sub simplicial category of $\mathrm{Path}[\Delta^n]$ with:

- $\mathrm{Ob}(\mathrm{Path}[\Lambda^n_j]) = \mathrm{Ob}(\mathrm{Path}[\Delta^n])$

- $\mathrm{Hom}_{C[\Lambda^n_j]}(i,k) = P_{i,k}$ unless $\begin{cases} i=0 \\ k=n \end{cases}$

- $\mathrm{Hom}_{C[\Lambda^n_j]}(0,n)$ is the subsimplicial set of $(\Delta^1)^{n-1}$ obtained by

 deleting the interior and the j-th face of the cube.

$\square$

I will not give the proof as it would take too much time, but the key steps are:

$*$ Λ^n_j is the pushout of the faces $\Delta^{[n]\setminus i}$ for $i \neq j$.

$\Rightarrow$ Formula for $\mathrm{Path}[\Lambda^n_j]$ in terms of $\mathrm{Path}[\Delta^{[n]\setminus i}]$

(Path commutes with colim.)

$*$ $P^{[n]\setminus i}_{k,\ell} \simeq P^{[n]}_{k,\ell}$ as soon as $k > i$ or $\ell < i$.

$*$ $P^{[n]\setminus i}_{0,n} \subseteq P^{[n]}_{0,n}$ is the subposet of those $I \subseteq [n]$ with $i \notin I$.

proof of Thm 19:

Let C be a locally Kan simplicial category.

and $0<j<n$. We must solve all lifting problems of the form

$$\begin{tikzcd} \Lambda^n_j \arrow[r, "f"] \arrow[d, hook] & N_\Delta(C) \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \quad \underset{\text{adj.}}{\Longleftrightarrow} \quad \begin{tikzcd} \mathrm{Path}[\Lambda^n_j] \arrow[r, "g"] \arrow[d] & C \\ \mathrm{Path}[\Delta^n] \arrow[ur, dashed] & \end{tikzcd}$$

To solve the problem on the right, it is enough by Lemma 21 to construct an extension in the induced diagram of simp. sets:

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Path}[\Lambda^n_j]}(0,n) \arrow[r] \arrow[d] & \mathrm{Hom}_C(g(0),g(n)) \\ \mathrm{Hom}_{\mathrm{Path}[\Delta^n]}(0,n) \arrow[ur, dashed] & \end{tikzcd}$$

(Note: we are very lucky that we only have to deal with <u>one</u> hom - simp. set !)

. By assumption, $\mathrm{Hom}_C(g(0), g(n))$ is Kan,

so we are done if we can show

that

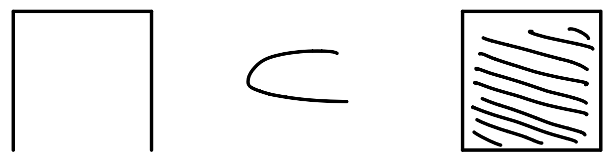
$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Path}[\Lambda^n_j]}(0,n) & \longrightarrow & \mathrm{Hom}_{\mathrm{Path}[\Lambda^n_j]}(0,n) \\ \text{Lemma 21} \;\; \text{IS} & & \text{IS} \\ (\Delta^1)^{n-1} \smallsetminus \left\{ \begin{matrix} \text{interior,} \\ j\text{-th face} \end{matrix} \right\} & \subseteq & (\Delta^1)^{n-1} \end{array}$$

is anodyne. (i.e in the saturated class generated by horn inclusions)

. This can be done by hand ; we are going to cheat and use a fact from simplicial homotopy theory (which I have mentioned already):

<u>Fact:</u> | A monomorphism $i: A \hookrightarrow B$ in sSet is anodyne <u>iff</u> $|i|: |A| \to |B|$ is an homotopy equivalence.

· The geometric realisation of the previous map is



which is clearly an homotopy equivalence.

This concludes the proof. $\square$

· Now we have our bridge between the two theories:

$$N_\Delta : \mathrm{Cat}_\Delta^{\mathrm{Kan}} \longrightarrow \mathrm{Cat}_\infty$$

Unfortunately, the left adjoint does not work in the same way: it is not true that if C is an ∞-category, then $\mathrm{Path}[C]$ is locally Kan.

Nevertheless we have:

Thm 22: (Lurie)

* Let $F : C \to D$ be a functor of locally Kan simplicial categories. Then

$$F \text{ is a Dwyer-Kan equivalence} \iff N_\Delta(F) \text{ is an equivalence of } \infty\text{-categories.}$$

* Let C be a locally Kan simplicial category. Then

$$\mathrm{Path}[N_\Delta(C)] \longrightarrow C$$

is a Dwyer-Kan equivalence. $\square$

. This (difficult) theorem is the core part of a more comprehensive comparison result between ∞-categories and simplicial categories: a Quillen equivalence of model categories. To even state it, we need to at least define models categories, which we will do soon.

. But first, let's use N_Δ to construct new examples of ∞-categories.

* For any cartesian closed category V, V is "enriched over itself": it is the underlying category of a V-enriched category $\tilde{V}$ with

$$\mathrm{Hom}_{\tilde{V}}(X,Y) = \underline{\mathrm{Hom}}_V(X,Y) \in V.$$

Composition is defined using the adjunction $- \times X \dashv \underline{\mathrm{Hom}}(X,-)$, as we saw in the case of sSet.

* Let $X \in$ sSet, K Kan complex. Then $\mathrm{Fun}(X,K)$ is also a Kan complex. This follows from similar arguments to the case of ∞-categories (see [Kerodon, Cor 3.1.3.4 (Tag 00TN)])

* We define $\widetilde{\mathrm{Kan}}$ to be the full simplicial subcategory of $\widetilde{\mathrm{sSet}}$ whose objects are Kan complexes. By the previous paragraph, $\widetilde{\mathrm{Kan}}$ is locally Kan.

<u>def 23</u> The ∞-category of spaces (or ∞-groupoids, or homotopy types, or anima) is the homotopy coherent nerve of $\widetilde{\mathrm{Kan}}$:

$$\boxed{\mathrm{Spc} := N_{\Delta}(\widetilde{\mathrm{Kan}}).}$$

* The ∞-category Spc plays a central role in the theory, analoguous to Set in 1-category theory.

* The above definition is only one of

several possible options. For instance, as the name suggests, there is a definition starting from topological spaces (CW-complexes, or a convenient category of top. spaces). The resulting quasicategories are then equivalent.

* There are even definitions of Spc which do not rely on a different model, like in Cisinski's book.

Nevertheless def 23 is probably the easiest, and lets you use already existing techniques from simplicial homotopy theory.

4) Differential graded categories

* Recall that a differential graded category (dg-category for short) is a category enriched over Cpl_R for some ring R.

- $\mathrm{Ob}(\mathrm{Cpl}_R)$: (C_*, ∂) with $\begin{cases} C_n \in \mathrm{Mod}_R \\ \partial_n : C_n \to C_{n-1} \\ \text{with } \partial_n \circ \partial_{n+1} = 0 \end{cases}$

- $\mathrm{Cpl}_R((C_*, \partial), (D_*, \partial)) = \left\{ f_n : C_n \to D_n \;\middle|\; \begin{tikzcd} C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\ D_n \arrow[r, "\partial_{n-1}"] & D_{n-1} \end{tikzcd} \right\}$

- $$(C_* \otimes D_*)_n := \bigoplus_{n = n' + n''} C_{n'} \otimes D_{n''}$$

and $$\partial(\underset{\substack{\cap \\ C_{n'}}}{c} \otimes \underset{\substack{\cap \\ D_{n''}}}{d}) := \partial(c) \otimes d + (-1)^{n'} c \otimes \partial(d)$$

* dg-categories are very common in homological algebra and algebraic geometry, in the same way that simplicial and topological categories are in algebraic topology.

* We write Cat^{dg}_R for the category of R-linear dg-categories.

4) Model categories